On Uniqueness for some non-Lipschitz SDE

Aureli Alabert
Department of Mathematics
Universitat Autònoma de Barcelona
08193 Bellaterra, Catalonia
Aureli.Alabert@uab.cat

Jorge A. León
Department of Automatic Control
Cinvestav-IPN
Apartado Postal 14-740
07000 México D.F, Mexico
jleon@ctrl.cinvestav.mx

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Abstract

We study the uniqueness in the path-by-path sense (i.e. ω-by-ω) of solutions to stochastic differential equations with additive noise and non-Lipschitz autonomous drift. The notion of path-by-path solution involves considering a collection of ordinary differential equations and is, in principle, weaker than that of a strong solution, since no adaptability condition is required. We use results and ideas from the classical theory of ode's, together with probabilistic tools like Girsanov's theorem, to establish the uniqueness property for some classes of noises, including Brownian motion, and some drift functions not necessarily bounded nor continuous.

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1 Introduction

Consider the stochastic differential equation (sde)

$$X_t = x_0 + \int_0^t b(X_s)\,ds + W_t\ , \quad t \in [0,T]\ , \tag{1}$$

where W is some noise process with continuous paths. That means, W is a random variable defined on some complete probability space $(\Omega, \mathcal{F}, P)$ with values in the space $C([0,T])$ of real continuous functions on $[0,T]$, endowed with its Borel σ-field. A canonical example is Brownian motion. The function $b\colon \mathbb{R} \to \mathbb{R}$ is supposed to be measurable at least, and x_0 is a given real number. We refer the reader to Karatzas and Shreve [14] for the concepts on stochastic processes that we use in this paper.

We recall that a *strong solution* of the equation above is a stochastic process X, with measurable paths, adapted to the filtration generated by W, and such that, for every t, the random variable $X_t - x_0 - \int_0^t b(X_s)\,ds$ is well defined and is equal to W_t almost surely. In fact, the form of the equation implies that X must have also continuous paths, hence the processes $\{X_t - x_0 - \int_0^t b(X_s)\,ds,\ t \in [0,T]\}$ and $\{W_t,\ t \in [0,T]\}$ will be indistinguishable, i.e. they will be equal as $C([0,T])$-valued random variables. It makes sense also to speak about local solutions, where the process X exists only up to some (random) time τ.

Uniqueness of solutions for the sde's (1) (sometimes called *strong uniqueness* or *pathwise uniqueness*) means that given a probability space, a process with the law of W defined on it, and the initial condition x_0, two strong solutions are indistinguishable. The classical existence and uniqueness result for sde of the type (1) is the following (see, e.g. [14, Theorems 5.2.5 and 5.2.9]):

Theorem 1.1. *If b is a Lipschitz function, then there is a unique strong solution to (1).*

Existence and uniqueness of a strong solution can be proved under much weaker conditions on b, at least for the case of a Brownian motion W. Indeed, it was shown by Veretennikov [20] that it is enough that b be bounded and measurable, also under some non-additive noises. This type of result was extended to parabolic differential equations in one space dimension driven by a space-time white noise by Bally, Gyöngy and Pardoux [3], Gyöngy [10] and Alabert and Gyöngy [2]. In the latter, as well as in Gyöngy and Martínez [11] in $\mathbb{R}^d$, the drift b is allowed to be locally unbounded, provided a suitable integrability condition holds. We refer the reader to Flandoli [8, Chapter 2] for a more complete discussion on the topic. For processes other than Brownian Motion, we can mention Nualart and Ouknine [16], [17]. We cite also Catellier and Gubinelli [5], where a slightly different problem is considered: The coefficient b is generalized to non-functions, that means, to distributional fields, leading to delicate problems about the meaning of the composition $b(X)$ and the definition of solution itself.

Now we introduce an ordinary differential equation similar to (1): Given a real continuous function $\omega \in C([0,T])$, we may write

$$x_t = x_0 + \int_0^t b(x_s)\,ds + \omega_t\ , \quad t \in [0,T]\ . \tag{2}$$

If b is a function for which the existence of a strong solution of the related sde (1) has been stated, one can say immediately that there exists a solution to (2) for almost all continuous functions ω with respect to the law of W. Nothing can be said of any particular ω, however.

Assume, on the other hand, that we could prove the existence of a solution to (2) for a certain class of functions ω having probability one with respect to the law of W. Would this yield an existence theorem for the sde? This is not clear, since the condition of adaptability in the definition of strong solution need not be satisfied, in principle.

According to Flandoli [8], we will call *path-by-path solution* of the sde (1) a solution obtained by solving ω-by-ω the corresponding class of ode's (2). Existence of a strong solution implies existence of a path-by-path solution, but the converse is not known to be true, in general. Similarly, uniqueness of the path-by-path solution does imply uniqueness in the strong sense, but not the other way round.

We ask ourselves if this gap can always be closed or, on the contrary, if it is possible to find counterexamples. This seems to be a difficult problem. Notice that in the classical case (b Lipschitz), it is true that a path-by-path solution is also strong. This is due to the Picard iteration scheme, which implies the existence of a strong solution, and to Gronwall's lemma, which gives the uniqueness in the path-by-path sense. Therefore, the question concerns only the non-Lipschitz cases.

We insist in the fact that establishing existence and uniqueness for a fixed particular $\omega \in C([0,T])$ is a different problem. For example, if $b(x) = \sqrt{|x|}$, $x_0 = 0$ and $\omega \equiv 0$, it is easy to see that the equation has exactly two local solutions (infinitely many, in a global sense), namely $x \equiv 0$ and $x_t = t^2/4$. However, the corresponding stochastic equation with a Brownian Motion W has a unique strong solution (see, e.g. [11]). Our results in Section 2 show in particular that the solution is unique also in the path-by-path sense.

Concerning uniqueness of path-by-path solutions, we only know the works of Davie [6, 7] and the remarks on them made by Flandoli [9]. In [6] it is proved, by means of an estimate quite complicated to obtain, that for a bounded measurable function b there is a unique solution to (2), for a class of continuous functions ω which has probability one with respect to the law of Brownian Motion. Hence, the solution to the corresponding sde, which was already known to exist in the strong sense, is not only strongly unique, but also path-by-path unique. In

[7] a diffusion coefficient is introduced, and the equation interpreted in the rough path sense. We provide a simpler proof of the path-by-path uniqueness in cases where b is not necessarily bounded or continuous; however, we have to restrict ourselves to dimension one, whereas in [6, 7] the equations are d-dimensional and the function b may also depend on time.

In this paper we apply some ideas from the theory of ordinary differential equations to study the path-by-path uniqueness of equation (1). Existence theorems are very general (e.g. Peano and Carathéodory theorems, that can be found in classical books like Hartman [12]); however, uniqueness (and non-uniqueness) results are poor and fragmented in comparison, and particularly scarce for equations of the form (2) (see, for instance, the book by Agarwal and Lakshmikantham [1], dedicated to the subject).

The paper is organized as follows. In Section 2 we use an extension of Iyanaga's uniqueness theorem (Theorem 2.3) for ode's, and Girsanov's theorem, to establish our main result: the path-by-path uniqueness of equation (1) for a Brownian motion W. Next, we see that the hypotheses on b can be relaxed if the noise has a constant sign, leading to similar theorems for the absolute value of the Brownian motion $|W_t|$ and for $-|W_t|$. In Section 3 we consider the particular case of the square root: Using Lakshmikantham's theorem (Lemma 3.4), we obtain a simpler proof when $b(x) = \sqrt{|x|}$ and the noise is non-negative; moreover, a discontinuous version of the square root exemplifies that continuity is not essential for the techniques of Section 2 to work. Finally, in Section 4 we use the idea behind the proof of Peano's uniqueness theorem to deal with some differentiable noises.

2 Main results

Let $\omega\colon [0,T] \to \mathbb{R}$ be a fixed continuous function, with $\omega_0 = 0$, and $b\colon \mathbb{R} \to \mathbb{R}$ a measurable function. Consider the equation

$$x_t = \int_0^t b(x_s)\, ds + \omega_t \ , \quad t \in [0,T] \ . \tag{3}$$

Taking $y_t := x_t - \omega_t$ as a new unknown function, (3) is equivalent to

$$y_t = \int_0^t b(y_s + \omega_s)\, ds \ , \quad t \in [0,T] \ . \tag{4}$$

Let us assume that this equation have at least one continuous solution $y\colon [0,T] \longrightarrow \mathbb{R}$, which is true by the Peano existence theorem if b is continuous (see, for instance, Lakshmikantham and Leela [15, Theorem 1.1.2]).

We will find some sufficient conditions on ω ensuring the uniqueness of that solution. Towards this end, consider the following set of hypotheses on function b:

H1. $b(0) = 0$.

H2. b is non-decreasing on $(0,\infty)$.

H3. b is continuous on $[0,\infty)$, and of class C^1 with b' non-increasing on $(0,\infty)$.

H4. $b(|x|) \le b(-|x|)$.

H5. b is non-increasing on $(-\infty, 0]$.

Notice that, under these hypotheses, any solution y of (4) is non-negative and non-decreasing. We will make use of the following two lemmas:

Lemma 2.1. *Assume hypotheses H1-H5 hold true. Then, for any two solutions y and $\bar{y}$ of equation (4), such that $y \le \bar{y}$, and for any continuous function ω with $\omega_0 = 0$ and not identically zero on the interval $[0,T]$, we have the inequality*

$$\begin{aligned} b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) &\le (\bar{y}_t - y_t) \cdot b'(|y_t + \omega_t|+) \\ &\le (\bar{y}_t - y_t) \cdot \Big[\mathbf{1}_{\{\omega_t \ge 0\}} \cdot b'\Big(\Big(\omega_t + \int_0^t \mathbf{1}_{\{\omega_s > 0\}} b(\omega_s)\, ds\Big)+\Big) \\ &\qquad + \mathbf{1}_{\{\omega_t < 0\}} \cdot b'(|y_t + \omega_t|+)\Big] \ , \quad t \in [0,T] \ , \end{aligned} \tag{5}$$

where $b'(z+)$ means $\lim_{x \downarrow z} b'(x)$, and $b'(0+)$ may be infinite.

Remark: We write right-limits only because b' is not necessarily defined at zero.

Proof. The proof will be divided into several cases.

Case 1: t such that $\omega_t \ge 0$.

By the mean value theorem, and using that b' is non-increasing on the positive axis,

$$b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) \le (\bar{y}_t - y_t) \cdot b'((y_t + \omega_t)+) \ ,$$

which, together with (4), implies

$$\begin{aligned} b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) &\le (\bar{y}_t - y_t) \cdot b'\Big(\Big(\omega_t + \int_0^t b(y_s + \omega_s)\, ds\Big)+\Big) \\ &\le (\bar{y}_t - y_t) \cdot b'\Big(\Big(\omega_t + \int_0^t \mathbf{1}_{\{\omega_s > 0\}} b(y_s + \omega_s)\, ds\Big)+\Big) \\ &\le (\bar{y}_t - y_t) \cdot b'\Big(\Big(\omega_t + \int_0^t \mathbf{1}_{\{\omega_s > 0\}} b(\omega_s)\, ds\Big)+\Big) \ . \end{aligned}$$

Case 2: t such that $-y_t \le \omega_t < 0$.

Similar to the case 1, the mean value theorem gives

$$b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) \le (\bar{y}_t - y_t) \cdot b'(|y_t + \omega_t|+) \ .$$

Case 3: t such that $-\frac{y_t + \bar{y}_t}{2} \le \omega_t < -y_t$.

We have $\bar{y}_t + \omega_t \ge -y_t - \omega_t > 0$. Therefore,

$$\begin{aligned} b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) &\le b(\bar{y}_t + \omega_t) - b(-y_t - \omega_t) \\ &\le (\bar{y}_t + y_t + 2\omega_t) \cdot b'(|y_t + \omega_t|+) \le (\bar{y}_t - y_t) \cdot b'(|y_t + \omega_t|+) \ , \end{aligned}$$

where we have used hypothesis H4 in the first inequality.

Case 4: t such that $-\bar{y}_t \le \omega_t < -\frac{y_t + \bar{y}_t}{2}$.

Now, $y_t + \omega_t < -\bar{y}_t - \omega_t \le 0$. Using again hypothesis H4,

$$b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) \le b(\bar{y}_t + \omega_t) - b(-y_t - \omega_t) \le 0 \ .$$

Case 5: t such that $\omega_t < -\bar{y}_t$.

Here, we have $y_t + \omega_t \le \bar{y}_t + \omega_t < 0$. Hence, by H5,

$$b(\bar{y}_t + \omega_t) - b(y_t + \omega_t) \le 0 \ .$$

□

Lemma 2.2. *Let $f, g, h\colon [0,T] \to \mathbb{R}$ be continuous functions, and $k\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ measurable. Assume*

$$f(t) \leq h(t) + \int_0^t k(s, f(s))\, ds \quad \text{and} \quad g(t) \geq h(t) + \int_0^t k(s, g(s))\, ds\ , \quad t \in [0,T]\ ,$$

and that k is non-decreasing in the second variable and

$$k(t,\bar{y}) - k(t,y) \leq a(t)(\bar{y} - y)\ , \quad \text{for } y, \bar{y} \in \mathbb{R},\ y \leq \bar{y}\ .$$

for some integrable function $a\colon [0,T] \to \mathbb{R}$.

Then $f(t) \leq g(t)$ for all $t \in [0,T]$.

Proof. See Pachpatte [18, Theorem 2.2.5]. □

Theorem 2.3. *Let b satisfy hypotheses H1-H5 and y be a solution of (4). Assume that the function*

$$a(t) = \Big[\mathbf{1}_{\{\omega_t \geq 0\}} \cdot b'\Big(\Big(\omega_t + \int_0^t \mathbf{1}_{\{\omega_s > 0\}} b(\omega_s)\, ds\Big)+\Big) + \mathbf{1}_{\{\omega_t < 0\}} \cdot b'(|y_t + \omega_t|+)\Big] \tag{6}$$

belongs to $L^1([0,T])$. Let $\bar{y}$ be another solution, with $y \leq \bar{y}$. Then, $y = \bar{y}$.

Remarks 2.4.

1. *If b is continuous, so that maximal and minimal solutions exist, one may say that a solution y satisfying Condition (6) is maximal. And that, if the minimal solution satisfies (6), then the solution to (4) is unique.*
2. *This result could be seen as an extension of Iyanaga's uniqueness theorem, where the function $a(t)$ was assumed to be continuous. For details see [1, Theorem 1.13.1].*
3. *We could replace H3-H5 by the existence of a measurable non-negative function g such that inequality (5) holds with $g(z)$ in place of $b'(z+)$, and write $g(|y_t + \omega_t|)$ in place of $a(t)$ in (6).*

Proof. Set $\phi = \bar{y} - y$ and $k(t,z) = a(t)z$, for $t \in [0,T]$ and $z \in \mathbb{R}$. Suppose that there is $t_1 \in (0,T)$ such that $\phi_{t_1} = z_1 > 0$ and consider the function u such that

$$\begin{cases} u(t) = z_0 + \int_0^t k(s,u(s))ds\ , & t \in [0,T] \\ u(t_1) = z_1\ . \end{cases}$$

Note that the definition of k yields that $z_0 > 0$. Hence, from Lemma 2.1, we have

$$\phi_t = \bar{y}_t - y_t = \int_0^t \left[b(\bar{y}_s + \omega_s) - b(y_s + \omega_s)\right] ds \leq \int_0^t k(s,\phi_s)ds\ , \quad t \in [0,T]\ ,$$

and therefore

$$\phi_t - u(t) \leq -z_0 + \int_0^t k(s, \phi_s - u(s))\, ds\ , \quad t \in [0,T]\ .$$

Thus, Lemma 2.2 applied to $f(t) := \phi_t - u(t)$, $g(t) := -z_0 \exp\{\int_0^t a(s)\, ds\}$ and $h(t) := -z_0$, leads to write

$$\phi_t - u(t) \leq -z_0 \exp\left\{\int_0^t a(s)ds\right\}\ , \quad t \in [0,T]\ ,$$

which, for $t = t_1$, gives $0 \leq -z_0 \exp\left\{\int_0^{t_1} a(s)ds\right\} < 0$, a contradiction. Consequently, $\phi \equiv 0$ on $[0,T]$ and the proof is complete. □

We use Theorem 2.3 in the proof of Theorem 2.6 below in order to show the path-by-path uniqueness for equation (4). Davie [6] computes a difficult estimate of the moments of the integral

$$\int_0^T (b(W_t + x) - b(W_t))\, dt \ .$$

to replace Lipschitz-type conditions in the study of a multidimensional version of (4) with bounded b. The use of direct results on ordinary differential equations allows a different and shorter proof in dimension one, valid for unbounded coefficients b. In the proof, besides Theorem 2.3, we make use of the following comparison theorem (see, for instance, Hartman [12, Theorem III.4.1]).

Lemma 2.5. *Let $h\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function, $c\in\mathbb{R}$ and u the minimal solution to*

$$u' = h(t,u), \qquad u_0 = c \ .$$

Also let v be a differentiable function such that $v_0 \geq c$ and $v'_t \geq h(t,v_t)$, $t\in[0,T]$. Then, on the interval of existence of u, $v_t \geq u_t$.

The following is the main result of this section.

Theorem 2.6. *Let W be a Brownian Motion on some probability space $(\Omega,\mathcal{F},P)$, and let $b\colon\mathbb{R}\to\mathbb{R}$ be a function satisfying hypotheses* H1-H5 *and:*

H6. *b is continuous and $|b(x)| \leq C(1+|x|)$, $\forall x$.*

H7. $\mathrm{E}_P\left[\int_0^T b'(|W_s|+)\, ds\right] < \infty$.

Then, the stochastic differential equation

$$X_t = \int_0^t b(X_s)\, ds + W_t \ , \quad t\in[0,T] \ , \tag{7}$$

has a unique path-by-path solution.

Proof. Notice that, since b is continuous, there exist minimal and maximal solutions to Equation (7) for every continuous path ω of the Brownian motion W. First, we are going to construct a solution adapted to the natural filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ of W which coincides with the minimal solution to (7); secondly, we will see that this adapted solution also coincides with the maximal solution. In conclusion, we will get the path-by-path uniqueness of the given stochastic differential equation.

Let $b_n(x) := b(x) - \frac{1}{n}$, $n\geq 1$. Consider a polynomial $p_n(x)$ such that

$$|b_n(x) - p_n(x)| < \varepsilon_n \ , \quad \text{for } x\in[-n,n] \ ,$$

with $\varepsilon_n = \frac{1}{2n(n+1)}$, and extend it as $p_n(x)\equiv p_n(n)$, for $x\geq n$, and $p_n(x)\equiv p_n(-n)$, for $x\leq -n$.

Let $f(t,y) := b(y+\omega_t)$, and $f_n(t,y) := p_n(y+\omega_t)$. The functions f_n are bounded, continuous and globally Lipschitz in the second variable, uniformly in the first. Therefore, the stochastic differential equation $Y^n_t = \int_0^t f_n(s,Y^n_s)\, ds$ (equivalently, $X^n_t = \int_0^t p_n(X^n_s)\, ds + W_t$) has a unique $\{\mathcal{F}_t\}$-adapted solution, which is a path-by-path solution of the corresponding deterministic equation for almost all Brownian sample paths. Also, $-2\leq f_n\leq f$ and f_n converges to f pointwise and monotonically from below.

By Lemma 2.5, applied to

$$\begin{cases} y_t^{n\prime} = f_n(t, y_t^n) \\ y_t' = f(t, y_t) \geq f_n(t, y_t) \\ y_0^n = y_0 = 0 \ , \end{cases}$$

where y is any solution of (4), we get $y \geq y^n$, on $[0,T]$. By the same comparison argument, since $\{f_n\}_n$ is non-decreasing, the sequence of solutions y^n is non-decreasing as $n \to \infty$.

Clearly, there exists a compact set $K \subset \mathbb{R}$ such that $y^n \colon [0,T] \to K$, for all n. Hence, by Dini's theorem, the sequence f_n converges uniformly to f when all functions are considered on $[0,T] \times K$.

We can then apply Theorem 1.2.4 of Hartman [12], which states that a certain subsequence y^{n_k} is uniformly convergent on $[0,T]$ to a solution of $y' = f(t,y)$. But since $\{y^n\}$ is increasing, it must itself converge to that solution. Finally, given that y^n is bounded from above by any solution of (4), the limit must be the minimal solution.

The stochastic process Y_t constructed in this way is therefore a solution, $\{\mathcal{F}_t\}$-adapted, of the stochastic differential equation $Y_t = \int_0^t b(Y_s + W_s)\, ds$. Hence, the process $X_t := Y_t + W_t$ is a strong solution of (7).

For the second part of the proof, we start with the process X just constructed, and prove first that $\int_0^T b'(|X_s|+)\, ds$ is finite almost surely. Indeed, let $\bar{X}$ be a Brownian motion under some other probability Q on $(\Omega, \mathcal{F})$; thanks to the linear growth condition H6 and [14, Corollary 3.5.16], Girsanov's theorem can be applied and there exists a probability $\bar{P}$ equivalent to Q such that $\bar{X}_t - \int_0^t b(\bar{X}_s)\, ds =: \bar{W}_t$ is a $\bar{P}$-Brownian motion. That means that $(\bar{X}, \bar{W})$ is a weak solution of equation (7).

By H7, and the equivalence of $\bar{P}$ and Q, we obtain

$$\bar{P}\Big(\int_0^T b'(|\bar{X}_s|+)\, ds < \infty \Big) = 1 \ . \tag{8}$$

The processes X and $\bar{X}$ have a.s. continuous paths under probabilities P and Q (hence $\bar{P}$), respectively. Therefore, $P\big(\int_0^T b(X_s)^2\, ds < \infty \big) = 1$ and $\bar{P}\big(\int_0^T b(\bar{X}_s)^2\, ds < \infty \big) = 1$, thanks to H6. Applying [14, Proposition 5.3.10], we obtain that the laws of the vector processes (X, W) under P and $(\bar{X}, \bar{W})$ under $\bar{P}$ are the same.

Consider the space of continuous functions $C([0,T])$ with its Borel σ-field and the $\mathbb{R}$-valued functional on $C([0,T])$ given by $\Lambda_g(x) := \int_0^T g(|x_s|)\, ds$, with $g \colon \mathbb{R}^+ \to \mathbb{R}$ continuous. Λ_g is a continuous functional, and therefore measurable. This is also true when g is the indicator function of an interval, due to the dominated convergence theorem. By the usual monotone class argument, we get that Λ_g is measurable for all bounded measurable functions g. And by the monotone convergence of the integrals, we get the same also for unbounded non-negative functions. That means that the law of the random variable $\Lambda_g(X)$ under P coincides with that of $\Lambda_g(\bar{X})$ under $\bar{P}$, and in particular, applied to $g(z) := b'(|z|+)$,

$$P\Big(\int_0^T b'(|X_s|+)\, ds < \infty \Big) = \bar{P}\Big(\int_0^T b'(|\bar{X}_s|+)\, ds < \infty \Big) \ ,$$

which together with (8) yields that $\int_0^T b'(|X_s|+)\, ds$ is a.s. finite, as we wished to see. It means that the process $Y_t = X_t - W_t$ satisfies that $t \mapsto b'(|Y_t + W_t|+)$ is a.s. integrable on $[0,T]$. Moreover, using hypotheses H3 and H7, the almost sure integrability of

$$\mathbf{1}_{\{\omega_t \geq 0\}} \cdot b'\Big(\Big(\omega_t + \int_0^t \mathbf{1}_{\{\omega_s > 0\}} b(\omega_s)\, ds \Big) + \Big)$$

is immediate. Applying Theorem 2.3 one concludes that $Y_t(\omega)$ coincides a.s. with the maximal solution to the deterministic equation (4).

We have seen that $Y_t(\omega)$ is both the minimal and maximal solution of (4). Hence, $X_t = Y_t + W_t$ is the unique path-by-path solution to the stochastic differential equation (7). □

Remark 2.7. *As a by-product of the proof, we have seen that under the conditions of the theorem, the unique path-by-path solution is also a strong solution, i.e. it is $\{\mathcal{F}_t\}$-adapted. Also, Theorem 2.6 is valid replacing b' by a function g satisfying the conditions of Remark 2.4(3) and hypothesis H7.*

Examples 2.8. The paradigmatic function satisfying all our hypothesis is the square root: $b(x) = \sqrt{|x|}$. In fact, all functions of the form $b(x) = |x|^\alpha$, with $0 < \alpha < 1$, are continuous non-Lipschitz functions satisfying H1-H7. To check H7, just notice that W_s is Gaussian with variance s, and therefore

$$\mathrm{E}[|W_s|^{\alpha-1}] = C_\alpha \cdot s^{\frac{\alpha-1}{2}} \ , \quad \text{for some constant } C_\alpha, \text{ for every } \alpha > 0 \ .$$

The hypothesis of continuity of b is needed in the proof of Theorem 2.6 to guarantee the existence of the uniform approximations of $b(x) - \frac{1}{n}$ on $[-n, n]$ by polynomials. However, one can allow for some discontinuities and ideas similar to those of the preceding proof can be applied. For example, this is the true with the function

$$b(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ \sqrt{-x} + 1, & \text{if } x < 0 \ . \end{cases}$$

We develop this particular case in the next section. □

Observe that the condition (6) for the uniqueness of solutions does not depend only on the noise function ω and the coefficient b, but also on the minimal solution y to (4). It is necessary to have an estimate of the type $b'(|y_t + \omega_t|+) \leq F(t)$, with an integrable function F, to obtain the uniqueness from Theorem 2.3. Hypothesis H6 was only used to this purpose. For a non-negative noise however, Condition (6) becomes

$$b'\Big(\Big(\omega_t + \int_0^t b(\omega_s)\, ds\Big)+\Big) \in L^1([0, T]) \tag{9}$$

and such an estimate is not necessary. Hypotheses H4 and H5 are not needed either. For instance, we can prove the following result, where the noise is the absolute value of a Brownian motion.

Proposition 2.9. *Let W_t be a Brownian Motion, and consider the stochastic differential equation*

$$X_t = \int_0^t b(X_s)\, ds + |W_t| \ , \quad t \in [0, T] \ . \tag{10}$$

Assume b satisfies hypotheses H1-H3 and H7. Then, equation (10) has at most one non-negative path-by-path solution, for almost all sample paths of W.

If, moreover, $b \geq 0$ on an interval $(-\varepsilon, 0)$, then this is the unique path-by-path solution.

Proof. As we have already pointed out, we only need to show that condition (9) holds true for almost all sample paths of $|W|$. But this is an easy consequence of the facts that $b(|W|)$ is non-negative, b' is non-increasing on $(0, \infty)$ and the expectation in H7 is finite. If b is also non-negative on an small interval to the left of 0, then any solution will be non-negative, and we get the path-by-path uniqueness. □

Remark 2.10. *One can also deduce from Condition (9) a result for negative noise: If* $\omega \leq 0$*, equation (3) is equivalent to*

$$z_t = \int_0^t \tilde{b}(z_s)\,ds + \tilde{\omega}_t$$

where $\tilde{b}(z) := -b(-z)$ *and* $\tilde{\omega}_t := -\omega_t$*. Therefore we obtain in this case the uniqueness of a non-positive solution under condition (9) and the hypotheses*

H1. $b(0) = 0$.

H2'. b *non-decreasing on* $(-\infty, 0)$.

H3'. b *continuous on* $(-\infty, 0]$*, and of class* C^1 *with* b' *non-decreasing on* $(-\infty, 0)$.

Hence, in this situation, the stochastic differential equation (10) with $-|W_t|$ *instead of* $+|W_t|$ *has a unique path-by-path non-positive solution; and it is the unique path-by-path solution if moreover* $b \leq 0$ *on some interval* $(0, \varepsilon)$.

Some known results for uniqueness in the theory of ordinary differential equations can be used in particular cases to obtain results similar to those above; this will be illustrated in the following sections. In Section 3 we consider the discontinuous case based in the square root that was mentioned in Examples 2.8, and the square root itself, $b(x) = |x|^{1/2}$, for a non-negative disturbance. For the latter, the results are not really better than applying the general setting above, but they are easier to obtain by other means. In Section 4, we study the uniqueness of the solution to equation (4) for some differentiable noises.

3 The particular case of the square root

3.1 Example: square root with a discontinuity

One can allow the function b to have some discontinuities and still get uniqueness of solutions. We illustrate this point with

$$b(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ \sqrt{-x} + 1, & \text{if } x < 0\ . \end{cases}$$

Defining

$$b_n(x) = \begin{cases} \sqrt{x} - \frac{1}{n}\ , & \text{if } x > 0 \\ \sqrt{-x} + 1 - \frac{1}{n}\ , & \text{if } x < -\frac{1}{n} \\ -(n + \sqrt{n})x - \frac{1}{n}\ , & \text{if } -\frac{1}{n} \leq x \leq 0\ , \end{cases}$$

we have that $\{b_n :\ n \in \mathbb{N}\}$ is a sequence of continuous functions on $\mathbb{R}$ such that, for $x \in \mathbb{R}$,

$$b_n(x) \leq b(x) \quad \text{and} \quad b_{n+1}(x) - b_n(x) \geq \frac{1}{n(n+1)}\ . \tag{11}$$

As in the proof of Theorem 2.6 we consider a polynomial p_n such that

$$|b_n(x) - p_n(x)| < \varepsilon_n\ , \quad \text{for } x \in [-n, n]\ ,$$

with $\varepsilon_n = \frac{1}{2n(n+1)}$, and extend it as $p_n(x) \equiv p_n(n)$, for $x \geq n$, and $p_n(x) \equiv p_n(-n)$, for $x \leq -n$. The definitions of b_n and p_n, together with (11), allow us to deduce that, for $x \in \mathbb{R}$,

$$p_{n+1}(x) - p_n(x) \geq 0 \quad \text{and} \quad b(x) \geq p_n(x) + \varepsilon_n\ . \tag{12}$$

Hence, $-\varepsilon_n - \frac{1}{n} \le p_n(x) < b(x)$, $x \in \mathbb{R}$. Therefore, using that b has linear growth, we can find a constant $K > 0$ such that

$$|p_n(x)| \le K(1 + |x|) \ , \quad \text{for} \ \ x \in \mathbb{R} \ \ \text{and} \ \ n \in \mathbb{N} \ . \tag{13}$$

Now we consider

$$Y_t^{(m)} = -\tilde{\varepsilon}_m + \int_0^t p_m(Y_s^{(m)} + W_s)\, ds \ , \quad t \in [0, T] \ , \tag{14}$$

where $\tilde{\varepsilon}_m \downarrow 0$ as $m \to \infty$, and W is a Brownian motion. Observe that the fact that p_m is a bounded Lipschitz function implies that equation (14) has a unique solution, which is measurable on $\Omega \times [0, T]$ and adapted with respect to the filtration $\{\mathcal{F}_t\}$ generated by W. In order to see that the minimal solution to equation (7) is also measurable and $\{\mathcal{F}_t\}$-adapted, we establish the following lemma.

Lemma 3.1. *Let Y be a solution of equation*

$$Y_t = \int_0^t b(Y_s + W_s)\, ds \ , \quad t \in [0, T] \ , \tag{15}$$

$m \in \mathbb{N}$ and $Y^{(m)}$ the solution of (14). Then,

$$Y_t \ge Y_t^{(m+1)} \ge Y_t^{(m)} \ , \quad t \in [0, T] \ .$$

Proof. By Lemma 2.5 and (12), we only need to see that $Y_t \ge Y_t^{(m)}$, for $t \in [0, T]$. By the continuity of Y and $Y^{(m)}$, and $\tilde{\varepsilon}_m > 0$, there is $t_0 \in (0, T]$ such that $Y_t > Y_t^{(m)}$, for $t \in [0, t_0]$. Now suppose that there exist $t_1 < T$ and $\eta > 0$ such that

$$Y_{t_1} = Y_{t_1}^{(m)} \quad \text{and} \quad Y_t^{(m)} > Y_t \ , \quad \text{for } t \in [t_1, t_1 + \eta] \ .$$

Then, for $h > 0$ small enough,

$$\frac{Y_{t_1+h}^{(m)} - Y_{t_1}^{(m)}}{h} = \frac{Y_{t_1+h}^{(m)} - Y_{t_1}}{h} > \frac{Y_{t_1+h} - Y_{t_1}}{h} \ .$$

Consequently,

$$p_m(Y_{t_1} + W_{t_1}) = p_m(Y_{t_1}^{(m)} + W_{t_1}) \ge D^+ Y_{t_1} \ ,$$

with $D^+ Y_{t_1} = \limsup_{h \downarrow 0} \frac{Y_{t_1+h} - Y_{t_1}}{h}$.

On the other hand, (12) leads to write

$$\frac{Y_{t_1+h} - Y_{t_1}}{h} = \frac{1}{h} \int_{t_1}^{t_1+h} b(Y_s + W_s)\, ds > \frac{1}{h} \int_{t_1}^{t_1+h} (p_m(Y_s + W_s) + \varepsilon_m)\, ds \ .$$

Therefore,

$$D^+ Y_{t_1} \ge p_m(Y_{t_1} + W_{t_1}) + \varepsilon_m > p_m(Y_{t_1} + W_{t_1}) \ ,$$

a contradiction. The proof is complete. □

Now we introduce the measurable and $\{\mathcal{F}_t\}$-adapted process $\bar{Y}_t := \lim_{m \to \infty} Y_t^{(m)}$, which is well-defined due to Lemma 3.1.

Lemma 3.2. *The process $\bar{Y}$ is absolutely continuous (i.e. it has absolutely continuous paths).*

Proof. Let $W_T^* = \sup_{t\in[0,T]} |W_t|$. Then (13) yields, for some constant K,

$$|Y_t^{(n)}| \le K \int_0^T |Y_s^{(n)}|\, ds + K(1 + T + TW_T^*) \ , \quad t \in [0,T] \ .$$

Thus, Gronwall's lemma implies

$$|Y_t^{(n)}| \le K(1 + T + TW_T^*) \exp(KT) \ , \quad t \in [0,T] \ . \tag{16}$$

It therefore follows that there exists a positive constant C such that, for $0 \le t_1 < t_2 < \ldots < t_\ell \le T$,

$$\sum_{i=1}^{\ell-1} |Y_{t_{i+1}}^{(n)} - Y_{t_i}^{(n)}| \le K \sum_{i=1}^{\ell-1} \int_{t_i}^{t_{i+1}} \Big(1 + |Y_s^{(n)} + W_s|\Big)\, ds \le C(1 + W_T^*) \sum_{i=1}^{\ell-1} (t_{i+1} - t_i) \ . \tag{17}$$

Finally, we prove the assertion of the lemma by letting $n \to \infty$. □

Observe that an immediate consequence of Lemma 3.2 and (17) (with $\ell = 2$) is that there exists a measurable and $\{\mathcal{F}_t\}$-adapted process A such that

$$\bar{Y}_t = \int_0^t A_s\, ds \quad \text{and} \quad |A_t| \le C(1 + W_T^*) \ , \quad t \in [0,T] \ . \tag{18}$$

Now we can state the main result of this example.

Theorem 3.3. *The process $\bar{Y}$ is the unique path-by-path solution of equation (15).*

Proof. We first observe that (18) and Girsanov's theorem (see Theorem 3.5.1 and Corollary 3.5.16 in Karatzas and Shreve [14]) imply that $W_t + \bar{Y}_t \neq 0$ for almost all $t \in [0,T]$, with probability 1. Now choose $s \in [0,T]$ so that $W_s + \bar{Y}_s \neq 0$ a.s. Then

$$\begin{aligned}
&\Big|p_n(Y_s^{(n)} + W_s) - b(\bar{Y}_s + W_s)\Big| \\
&\le \Big|p_n(Y_s^{(n)} + W_s) - b_n(Y_s^{(n)} + W_s)\Big| + \Big|b_n(Y_s^{(n)} + W_s) - b(Y_s^{(n)} + W_s)\Big| \\
&\quad + \Big|b(Y_s^{(n)} + W_s) - b(\bar{Y}_s + W_s)\Big| \\
&\le \varepsilon_n + \frac{1}{n} + \Big|b(Y_s^{(n)} + W_s) - b(\bar{Y}_s + W_s)\Big| \ .
\end{aligned}$$

So we can conclude that $p_n(Y_s^{(n)} + W_s) \to b(\bar{Y}_s + W_s)$ a.s. as $n \to \infty$ due to the continuity of b on $\mathbb{R} - \{0\}$. Hence, (13), (14) and (16) give

$$\bar{Y}_t = \int_0^t b(\bar{Y}_s + W_s) ds \ , \quad t \in [0,T] \ .$$

We have obtained that the $\{\mathcal{F}_t\}$-adapted process $\bar{Y}$ is, by Lemma 3.1, the minimal solution, a.s. Now we can finish as in the proof of Theorem 2.6. Instead of the continuity of b it is enough that b be locally bounded to use that $P\{\int_0^t b(X_s)^2\, ds < \infty\} = 1$. □

3.2 Example: Square root and non-negative noise

We assume in this section that $\omega\colon [0,T] \to [0,\infty)$ is a fixed continuous non-negative function. Consider the equation

$$x_t = \int_0^t \sqrt{|x_s|}\, ds + \omega_t \ , \quad t \in [0,T] \ . \tag{19}$$

and its equivalent, defining $y_t = x_t - \omega_t$,

$$y_t = \int_0^t \sqrt{|y_s + \omega_s|}\, ds \ , \quad t \in [0,T] \ . \tag{20}$$

Any solution y of (20) is clearly a continuously differentiable, positive and non-decreasing function. The absolute value inside the square root is therefore unnecessary.

We will make use of the following uniqueness theorem (see Agarwal and Lakshmikantham [1, Theorem 2.8.3]):

Lemma 3.4. *(Lakshmikantham's Uniqueness Theorem). Suppose that $f(t,y)$ is defined in $D := (0,T] \times [-a,a]$, measurable in t for each fixed y, continuous in y for each fixed t, and there exists an integrable function M on the interval $[0,T]$ such that $|f(t,y)| \le M(t)$ on D. Consider the ordinary one-dimensional differential equation*

$$y'(t) = f(t, y(t)) \ , \quad y(0) = 0 \ , \tag{21}$$

and define a classical solution *of (21) on $[0,T]$ as a function y satisfying the initial condition, continuous in $[0,T]$, and differentiable and verifying the equation on $(0,T]$.*

Assume that:

i) Any two classical solutions y and $\bar y$ of (21) satisfy

$$\lim_{t\to 0+} \frac{|\bar y_t - y_t|}{B_t} = 0 \ ,$$

where B is a continuous and positive function on $(0,T]$ with $\lim_{t\to0^+} B(t) = 0$.

ii) There is a continuous and non-negative function $g\colon (0,T]\times[0,2a]$ for which the only solution z of $z_t' = g(t,z)$ on $[0,T]$ such that $\lim_{t\to0^+} \frac{z_t}{B_t} = 0$ is the trivial solution $z \equiv 0$.

iii) f is defined on $\bar D$ (the closure of D), and for all (t,y) and $(t,\bar y)$ in D, the inequality $|f(t,\bar y) - f(t,y)| \le g(t, |\bar y - y|)$ is satisfied.

Then, equation (21) above has at most one classical solution on $[0,T]$.

Our equation reads $y_t' = f(t,y_t)$, with $f(t,y) = \sqrt{y+\omega_t}$. In this case, moreover, the function f is continuous, and Lakshmikantham theorem implies the uniqueness of ordinary solutions (i.e. of class C^1 in $[0,T]$).

Let y and $\bar y$ be the minimal and maximal solutions of (20), respectively. By the mean value theorem applied to $f(x) := \sqrt{x}$, we have

$$\sqrt{\bar y_t + \omega_t} - \sqrt{y_t + \omega_t} = \frac{\bar y_t - y_t}{2} \cdot (\omega_t + \xi_t)^{-1/2} \ , \quad t \in [0,T] \ ,$$

for some $\xi_t \in [y_t, \bar y_t]$. Since $\xi_t \ge y_t \ge \int_0^t \omega_s^{1/2}\, ds$, we find the bound

$$\sqrt{\bar y_t + \omega_t} - \sqrt{y_t + \omega_t} \le \frac{\bar y_t - y_t}{2} \cdot \Big(\omega_t + \int_0^t \omega_s^{1/2}\, ds\Big)^{-1/2} \ , \quad t \in [0,T] \ . \tag{22}$$

We also have that (20) implies

$$(\bar{y}_t - y_t)' \le \sqrt{\bar{y}_t - y_t}\ , \quad t \in [0,T]\ .$$

Using Lakshmikantham and Leela [15, Theorem 1.4.1], the difference $\bar{y}_t - y_t$ is bounded by the maximal solution to $z_t = \int_0^t \sqrt{z_s}\, ds$, which is $z_t = t^2/4$. Now, taking $B(t) = t^\alpha$, with any $\alpha \in (0,2)$, hypothesis (i) of Lemma 3.4 is clearly satisfied.

For conditions (ii) and (iii), notice that, by (22), we can take

$$g(t,z) := \frac{z}{2} \cdot \Big(\omega_t + \int_0^t \omega_s^{1/2}\, ds\Big)^{-1/2}, \quad t \in (0,T] \text{ and } z \in \mathbb{R}^+\ ,$$

assuming the expression on the right makes sense. The differential equation $z_t' = g(t, z_t)$ is linear, and all its solutions can be explicitly written as

$$z_t = c \exp\Big\{\frac{1}{2}\int_{t_0}^t \Big(\omega_s + \int_0^s \omega_r^{1/2} dr\Big)^{-1/2} ds\Big\}$$

for some constant c and $t_0 \in (0,T]$. Then, if $s \mapsto \omega_s + \int_0^s \omega_r\, dr$ is integrable at 0^+, those solutions can only tend to zero at the origin if $z \equiv 0$.

Thus, we have proved the following result.

Theorem 3.5. *Assume that the noise ω is such that $\Big(\omega_\cdot + \int_0^\cdot \omega_s^{1/2}\, ds\Big)^{-1/2} \in L^1([0,T])$. Then, there exists a unique solution to equation (19).*

As an immediate consequence of this theorem, we recover the result of Proposition 2.9 in an easier way:

Corollary 3.6. *Let W be a Brownian Motion, and consider the stochastic differential equation*

$$X_t = \int_0^t \sqrt{|X_s|}\, ds + |W_t|\ , \quad t \in [0,T]\ . \tag{23}$$

Then, equation (23) has a unique path-by-path solution, for almost all paths of W.

Proof. In view of Theorem 3.5, we only have to show that for almost all sample paths ω of a Brownian motion,

$$\Big(|\omega_\cdot| + \int_0^\cdot |\omega_s|^{1/2}\, ds\Big)^{-1/2} \in L^1([0,T])\ ,$$

and this has already been checked in Examples 2.8.

□

4 Differentiable noise

In this section we analyze the uniqueness of a solution to equation (3) for some differentiable perturbations. Equivalently, we are dealing with the absolutely continuous solutions to

$$\begin{cases} x_t' = b(x_t) + w_t'\ , & t\text{-a.e. on } [0,T] \\ x_0 = 0\ , \end{cases} \tag{24}$$

where ω is a function in $C^1([0,T])$, and we want to keep at a minimum the hypotheses on b.

We state first a general result for noises with a strictly negative derivative. We mimic the proof of Peano's uniqueness theorem (see, for instance, [1, Theorem 1.3.1]).

Theorem 4.1. *Let ω be a C^1 function on $[0,T]$ such that $\omega_0 = 0$ and with negative derivative (i.e. $\omega'_t < 0$ for $t \in [0,T]$).*

Assume that

i) b is measurable and $\lim_{x\to 0} b(x) = b(0) = 0$.

ii) For some $\eta > 0$, there exists an increasing continuous function $g\colon [0,\eta) \to \mathbb{R}$, of class C^1 on $(0,\eta)$, with g' non-increasing, and such that:

H8. *$x \mapsto g'(-x)b(x)$ is non-increasing on $(-\eta,0)$.*

iii) There exists either a maximal or a minimal solution to (24).

Then, there is a unique local solution to equation (24). Global uniqueness on $[0,T]$ is true if $\eta = \infty$.

Proof. Let x and $\bar{x}$ be two solutions such that $x \leq \bar{x}$. We first observe that for some $\varepsilon > 0$, we have $-\eta/2 \leq x, \bar{x} < 0$ on $(0,\varepsilon)$ due to the continuity of ω' and b at zero, and to $\omega'_0 < 0$.

Define $z_t := g(-x_t)$ and $\bar{z}_t := g(-\bar{x}_t)$ on $[0,\varepsilon)$. Both z and $\bar{z}$ are absolutely continuous since g is C^1 and x and $\bar{x}$ are absolutely continuous, and

$$z'_t = -g'(-x_t)(b(x_t) + \omega'_t) \quad \text{and} \quad \bar{z}'_t = -g'(-\bar{x}_t)(b(\bar{x}_t) + \omega'_t)\ , \quad t\text{-a.e. on } (0,\varepsilon)\ .$$

The fundamental theorem of calculus gives, for $0 < \delta < t < \varepsilon$,

$$\begin{aligned}(z_t - \bar{z}_t) - (z_\delta - \bar{z}_\delta) &= \int_\delta^t \big[-g'(-x_s)b(x_s) + g'(-\bar{x}_s)b(\bar{x}_s)\big]\,ds + \int_\delta^t -\omega'_s \cdot \big(g'(-x_s) - g'(-\bar{x}_s)\big)\,ds \\ &\leq \int_\delta^t -\omega'_s \cdot \big(g'(-x_s) - g'(-\bar{x}_s)\big)\,ds\ ,\end{aligned}$$

since, by H8, the first integral is non-positive. Letting $\delta \to 0$ and using *ii)*, we obtain $0 \leq z_t - \bar{z}_t \leq 0$. Consequently we have that $z = \bar{z}$ on $[0,\varepsilon)$. And, since g is increasing, we get $x = \bar{x}$ on $[0,\varepsilon]$.

Now assume that the function g is defined on $[0,\infty)$, and that uniqueness holds up to $t_0 < T$. Any solution x will satisfy

$$x_t = x_{t_0} + \int_{t_0}^t b(x_s)ds + \omega_t - \omega_{t_0}\ , \quad t \in [t_0, T]\ ,$$

where x_{t_0} is a common value to all of them. Notice that every time x hits the origin, it is differentiable at that point and its derivative is negative. Therefore, $x_{t_0} \leq 0$ and, by continuity, two solutions x and $\bar{x}$ will be negative in some interval $(t_0, t_0+\varepsilon)$. We can proceed again as in the beginning of the proof to extend uniqueness beyond t_0. □

Remarks 4.2.

1. *A well known sufficient condition for the existence of maximal and minimal solutions is the continuity of b. But weaker conditions exists in the literature. For instance, in the situation given, this is true if:*

   (a) b has linear growth, and

   (b) $\limsup_{y\to x^-} b(y) \leq b(x) \leq \liminf_{y\to x^+} b(y)$, for all x.

   These conditions follow easily from the general Theorem 3.1 in [13]. Even weaker conditions, allowing jumps in the "wrong direction", can be found in [19] and [4].

2. *The continuity of b at zero can be replaced by other conditions ensuring that the solutions remain negative. For example, if*

   (a) $\limsup_{x\to 0} b(x) \le 0$, *or*

   (b) *There is a non-decreasing continuous function f such that $b \le f$ on an open interval containing 0 and $f(0) + \omega_0' < 0$.*
   In this case,
$$x_t = \int_0^t (b(x_s) + \omega_s')\, ds \le \int_0^t (f(x_s) + \omega_s')\, ds$$
   and x is bounded by the maximal solution of
$$u_t = \int_0^t (f(u_s) + \omega_s')\, ds$$
   (see Pachpatte [18, Theorem 2.2.4]), which is negative on an interval $(0, \eta)$.

3. *An example where the above remarks apply is given by*
$$b(x) = \begin{cases} \sqrt{x}, & \text{if } x \ge 0 \\ \sqrt{-x} - 1, & \text{if } x < 0\ . \end{cases}$$
   A maximal solution exists by the sufficient conditions of statement 1. Then both (a) or (b) of statement 2 are applicable with $f(x) = \sqrt{x} \cdot \mathbf{1}_{\{x \ge 0\}}$ in the second case.

The following result is also inspired in the proof of Peano's uniqueness theorem. We consider a particular example of an ordinary differential equation driven by a differentiable noise, positive in a neighbourhood of zero, but changing sign afterwards. By "piecewise Lipschitz" below we mean a function whose domain can be partitioned into intervals such that their interior is non-empty and the function is Lipschitz on each of them.

Example 4.3. *Consider $\omega_t = \alpha t + t^{2+\beta} \sin(t^{-1})$, where α, $\beta > 0$, $t \in (0, T]$ and $\omega_0 = 0$.*

Assume:

i) b is measurable and $\lim_{x\to 0} b(x) = b(0) = 0$.

ii) For some $\eta > 0$, there exists an increasing continuous function $h\colon [0, \eta) \to \mathbb{R}$, of class C^1 on $(0, \eta)$, with h' non-increasing, and such that:

  H9. *$x \mapsto h'(x)b(x)$ is non-increasing on $(0, \eta)$.*

iii) There exists either a maximal or a minimal solution to (24).

Then, there is a unique local solution to equation (24). Global uniqueness on $[0, T]$ is true if, furthermore,

i') b is non-negative, either piecewise Lipschitz or locally Lipschitz on $(-\infty, 0)$, and locally Lipschitz on $(0, \infty)$.

ii') $\eta = \infty$.

iv) There exists $g\colon [0, \infty) \to \mathbb{R}$ satisfying assumption ii) of Theorem 4.1,

Then, equation (24) has a unique solution.

Proof. If x is a solution to (24) with the given noise ω, we can see, as in the preceding theorem, that there is an $\varepsilon > 0$ such that $0 < x < \eta/2$ and $\omega' > 0$ on $(0, \varepsilon)$. Given any two such solutions with $x \leq \bar{x}$, define $z_t := h(x_t)$ and $\bar{z}_t := h(\bar{x}_t)$, on $[0, \varepsilon)$.

Hence, proceeding as in the proof of Theorem 4.1 but using hypothesis H9 instead of H8, we obtain that

$$0 \leq \bar{z}_t - z_t \leq \int_0^t \omega'_s \cdot \left(h'(\bar{x}_s) - h'(x_s)\right) ds \leq 0 \ ,$$

and therefore equation (24) with the given ω has a unique solution on $[0, \varepsilon]$.

For the second part, assume that uniqueness holds up to $t_0 < T$. We distinguish the following cases:

Case 1: $x_{t_0} > 0$.

We only need to use that b is locally Lipschitz to extend the uniqueness to the right of t_0.

Case 2: $x_{t_0} \leq 0$, $\omega'_{t_0} < 0$.

Here we use condition H8, and we finish as in Theorem 4.1.

Case 3: $x_{t_0} \leq 0$, $\omega'_{t_0} \geq 0$.

We can write

$$\begin{aligned} \omega''_{t_0} &= \frac{1+\beta}{t_0}\omega'_{t_0} - (\beta+1)\frac{\alpha}{t_0} - (\beta+1)t_0^{\beta-1}\cos(t_0^{-1}) - t_0^{\beta-2}\sin(t_0^{-1}) \\ &\geq -(\beta+1)\frac{\alpha}{t_0} - (\beta+1)t_0^{\beta-1}\cos(t_0^{-1}) - t_0^{\beta-2}\sin(t_0^{-1}) \\ &\geq -(\beta+1)\frac{\alpha}{t_0} + (\beta+1)\left[\frac{-\alpha}{t_0} - (\beta+2)t_0^{\beta}\sin(t_0^{-1})\right] - t_0^{\beta-2}\sin(t_0^{-1}) \ . \end{aligned} \tag{25}$$

On the other hand, the facts that $x_{t_0} \leq 0$ and b is non-negative imply $\omega_{t_0} \leq 0$. Thus, $-t_0^{\beta+2}\sin(t_0^{-1}) \geq \alpha t_0 > 0$, which, together with (25), yields

$$\begin{aligned} \omega''_{t_0} &> (\beta+1)\left[-2\frac{\alpha}{t_0} - (\beta+2)t_0^{\beta}\sin(t_0^{-1})\right] \\ &\geq -(\beta^2+\beta)t_0^{\beta}\sin(t_0^{-1}) > 0 \ . \end{aligned}$$

Therefore, there exists $\varepsilon > 0$ such that $\omega' > 0$ on $(t_0, t_0+\varepsilon)$. If $x_{t_0} = 0$, we can proceed as in the beginning of this proof; if $x_{t_0} < 0$, and since b is non-negative, we have $x' > 0$ on $(t_0, t_0+\varepsilon)$. Therefore x_t is increasing and the piecewise Lipschitz property of b on $(-\infty, 0)$ gives the uniqueness beyond t_0, even if b is discontinuous at t_0.

□

Hypotheses H8 and H9 in Theorem 4.1 and Example 4.3 are satisfied by functions of the form

$$b(x) = \begin{cases} r_1(x) \cdot s_1(x) \ , & x \geq 0 \\ r_2(x) \cdot s_2(-x) \ , & x < 0 \ , \end{cases}$$

where r_1 and r_2 are non-negative and non-increasing, with r_2 piecewise locally Lipschitz, and s_1 and s_2 are positive, non-decreasing, continuous on $[0, \infty)$, with $1/s_1$ and $1/s_2$ integrable at zero. One can take $h(x) = \int_0^x 1/s_1$, and $g(x) = \int_0^x 1/s_2$. In particular, this family includes the non-Lipschitz functions $b(x) = |x|^\alpha$ $(0 < \alpha < 1)$, and, more generally, $b(x) = r(x) \cdot |x|^\alpha$, with convenient r; it suffices to take $g(x) = h(x) = \frac{1}{1-\alpha}x^{1-\alpha}$.

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References

[1] R. P. Agarwal and V. Lakshmikantham. *Uniqueness and nonuniqueness criteria for ordinary differential equations*, volume 6 of *Series in Real Analysis*. World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[2] Aureli Alabert and István Gyöngy. On stochastic reaction-diffusion equations with singular force term. *Bernoulli*, 7(1):145–164, 2001.

[3] V. Bally, I. Gyöngy, and É. Pardoux. White noise driven parabolic SPDEs with measurable drift. *J. Funct. Anal.*, 120(2):484–510, 1994.

[4] Daniel C. Biles and Rodrigo López Pouso. First-order singular and discontinuous differential equations. *Bound. Value Probl.*, pages Art. ID 507671, 25, 2009.

[5] R. Catellier and M. Gubinelli. On the regularization of ODEs via irregular pertubations. *arXiv:1205.1735 [math.PR]*, 2012.

[6] Alexander M. Davie. Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN*, (24):Art. ID rnm124, 26, 2007.

[7] Alexander M. Davie. Individual path uniqueness of solutions of stochastic differential equations. In *Stochastic analysis 2010*, pages 213–225. Springer, Heidelberg, 2011.

[8] Franco Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.

[9] Franco Flandoli. Regularizing properties of Brownian paths and a result of Davie. *Stoch. Dyn.*, 11(2-3):323–331, 2011.

[10] István Gyöngy. Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Process. Appl.*, 73(2):271–299, 1998.

[11] István Gyöngy and Teresa Martínez. On stochastic differential equations with locally unbounded drift. *Czechoslovak Math. J.*, 51(126)(4):763–783, 2001.

[12] Philip Hartman. *Ordinary differential equations*. John Wiley & Sons, Inc., New York-London-Sydney, 1964.

[13] Ezzat Ramadan Hassan and Witold Rzymowski. Extremal solutions of a discontinuous scalar differential equation. *Nonlinear Anal.*, 37(8, Ser. A: Theory Methods):997–1017, 1999.

[14] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[15] V. Lakshmikantham and S. Leela. *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations.* Academic Press, New York-London, 1969. Mathematics in Science and Engineering, Vol. 55-I.

[16] David Nualart and Youssef Ouknine. Regularization of differential equations by fractional noise. *Stochastic Process. Appl.*, 102(1):103–116, 2002.

[17] David Nualart and Youssef Ouknine. Regularization of quasilinear heat equations by a fractional noise. *Stoch. Dyn.*, 4(2):201–221, 2004.

[18] B. G. Pachpatte. *Inequalities for differential and integral equations*, volume 197 of *Mathematics in Science and Engineering.* Academic Press, Inc., San Diego, CA, 1998.

[19] Rodrigo L. Pouso. On the Cauchy problem for first order discontinuous ordinary differential equations. *J. Math. Anal. Appl.*, 264(1):230–252, 2001.

[20] A. Ju. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)*, 111(153)(3):434–452, 480, 1980.